Precalculus, Quarter 1, Unit 1.1

Analysis of Functions

Overview

Number of Instructional Days: 8 (1 day = 45 minutes)

Content to Be Learned

- Analyze different representations of functions to identify key features such as domain, range, zeros, intercepts, end behavior, extrema, concavity, asymptotes, intervals of increase and decrease, and constant behavior.
- Compose functions utilizing appropriate, accepted mathematical notation; identify key features; and identify domain of the composition.
- Verify by composition that one function is the inverse of another.
- Determine the inverse of a function, if it exists.
- Restrict the domain of an inverse function.

Mathematical Practices to Be Integrated

Make sense of problems and persevere in solving them.

- Explain correspondence among equations, verbal descriptions, tables, and graphs.
- Graph important features and relationships of functions.

Reason abstractly and quantitatively.

- Decontextualize vocabulary.
- Represent a given situation symbolically.
- Manipulate the notation to combine functions.

Essential Questions

- Given certain characteristics of the function, how would you determine the graphical representation of functions?
- Given any function, how would you algebraically and graphically determine the inverse of the function?
- What is an example of an equation of a function that is its own inverse?
- What is an example of an equation a function that does not have an inverse on the domain of $(-\infty,\infty)$ but does for the domain $(0,\infty)$?
- Given that $f(x)$ and $g(x)$ are odd functions, explain how you would determine if $(f \bullet g)(x)$ is odd, even, or neither?
- What equations would illustrate an example of a function where $f(x)=f(x+p)$?

Written Curriculum

Common Core State Standards for Mathematical Content

Interpreting Functions F-IF

Analyze functions using different representations

F-IF.7 Graph functions expressed symbolically and show key features of the graph~~, by hand in simple cases and using technology for more complicated cases~~.★

d. (+) Graph ~~rational~~ functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

Building Functions F-BF

Build a function that models a relationship between two quantities

F-BF.1 Write a function that describes a relationship between two quantities.★

c. (+) Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

Build new functions from existing functions

F-BF.4 Find inverse functions.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Common Core Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Clarifying the Standards

Prior Learning

Students learned, as general principle, the concept of a function and the use of function notation in algebra 1 (F.IF.1, 2, 3).

They interpreted linear, exponential, and quadratic functions that arose in applications in terms of a context (F.IF.4, 5, 6). They analyzed linear, exponential, quadratic, absolute value, step, and piece-wise functions using different representations in algebra 1 (F.IF.7a, 7b, 7e, 8a, 8b, 9), and interpreted and used key features to guide the selection of an appropriate type of model function in algebra 2 (F.IF.4, 5, 6, 7b, 7c, 7e, 8a, 8b, 9).

Students built new functions using the four basic operations from existing linear, exponential, quadratic, and absolute value functions in algebra 1 (F.BF.1a, 1b, 2, 3, 4a) and built new functions using the four basic operations from existing simple radical, rational, and exponential functions in algebra 2 (F.BF.1b, 3, 4a).

Current Learning

Given different representations of functions, students identify domain, range, zeros, intercepts, end behavior, extrema, concavity, asymptotes, intervals of increase and decrease, and constant behavior.

They identify and classify functions based on continuity and discontinuity as well as on symmetry and periodicity.

Students also compose new functions from given functions and analyze functions and their inverses.

Future Learning

Students will apply these concepts in studying polynomial, rational, exponential, logarithmic, and trigonometric functions, and their inverses, later in this course. They will also evaluate a limit of a function based on continuity, end behavior, and asymptotes later in this course. All these concepts will lay the foundation for calculus.

Additional Findings

The findings in this study indicated that students' mastering the connection between graphs and equations is not as straightforward as is often assumed. It is important for students to understand which representation for a function is best. However this cannot be where a teacher stops. Developing a thorough and deep understanding of functions also means being able to interchange and work flexibly between the different representations. To help students master these connections, it is important to give students opportunities to build connections between the algebraic and graphical representations through interacting with them. [Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), (pp. 500–507)].

According to *Benchmarks for Science Literacy*, studying algebra includes reflecting on relationships and patterns and their representation as data. Students have an opportunity to reflect on relationships in real-world patterns and to represent this relationship in numeric, graphical, analytical, and verbal modalities (pp. 23–25, 29, 214–216).

Teacher Notes:

How would you determine the graphical representation of functions, given characteristics of the function, such as the following:

1. Jump discontinuity at $x = -3$
2. As $x \to \infty$, $f(x) = 0$
3. Removable discontinuity at $x = 5$
4. Relative maximum of 7
5. As $x = +\infty$ the function is unbound.
6. As $x \to 2^{-}$, $f(x) \to +\infty$; as $x \to 2^{+}$, $f(x) \to -\infty$
7. On the interval (-5, -3), *f(x)* is increasing
8. Concave down from $(10, \infty)$

Challenges for students include:

- Paying attention to the specificity of notation and vocabulary in the analysis of functions (e.g., in looking at extrema, students should look at function value rather than the ordered pair).
- Determining domain for composition of functions.

Precalculus, Quarter 1, Unit 1.2

Polynomial and Rational Functions

Overview

Number of Instructional Days: 16 (1 day = 45 minutes)

Content to Be Learned

- Use the Rational Zero Theorem, upper and lower bounds, and the Intermediate Value Theorem to determine all roots of a polynomial equation.
- Apply DesCartes' Rule of Signs to determine the nature of the roots of a polynomial function.
- Apply complex conjugates and division by a complex number in problem situations.
- Determine a polynomial function having complex roots.
- Determine removable discontinuity, vertical, horizontal, and slant asymptotes of rational functions.
- Use multiplicity of the factors of the denominator of a rational function to determine function behavior at the vertical asymptotes.
- Given complex roots, points of discontinuity, and asymptotes, determine a rational function and graph it.
- Write and solve functions that are compositions of two other functions.

Mathematical Practices to Be Integrated

Make sense of problems and persevere in solving them.

- Graph important features and relationships of polynomial and rational functions.
- When interpreting different representations of a function, reflect on whether or not the work makes sense.

Look for and make use of structure.

- Understand complicated concepts, such as rational functions as single objects or as a composition of several objects.
- Having found the zeros and end behavior, discern a pattern or structure of the graph.

Look for and express regularity in repeated reasoning.

- When analyzing polynomial and rational functions, use all the general characteristics of the functions.
- When multiplying complex conjugates and rationalizing a complex denominator, recognize the repeated pattern of simplifying terms.

Essential Questions

- How do the characteristics of a function determine its equation?
- Given all roots, how would you create and interpret the equation of the polynomial function?
- Given a polynomial function and one of its complex roots with an imaginary unit, how would you determine the remaining roots?
- Given a graph of a polynomial or rational function, how do you find an equation for the graph?
- How does the linear factorization of the numerator/denominator of a rational function describe the graph of the function?
- How does the linear factorization of a polynomial function describe its graph?

Written Curriculum

Common Core State Standards for Mathematical Content

Interpreting Functions F-IF

Analyze functions using different representations

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

Building Functions F-BF

Build a function that models a relationship between two quantities

F-BF.1 Write a function that describes a relationship between two quantities.★

c. (+) Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.

N-CN.3 (+) Find the conjugate of a complex number; use conjugates ~~to find moduli and quotients~~ of complex numbers.

Use complex numbers and their operations on the complex plane.

N-CN.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x+2i)(x-2i)$.

Common Core Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Clarifying the Standards

Prior Learning

In algebra 1, students analyzed linear, exponential, quadratic, absolute value, step, and piece-defined functions using different representations, and they interpreted linear, exponential, and quadratic functions that arose in applications within a context. In algebra 2, students analyzed and interpreted the functions previously studied with an emphasis on using key features to select appropriate models (F.IF 4, 5, 6, 7a, 7b, 7E, 8a, 8b, 9).

Also in algebra 1, students built linear, exponential, and quadratic functions that model relationships between two quantities, and they built new functions using the basic operations from existing function types. Algebra 2 students built simple radical, rational, and exponential functions that model relationships, and they built new functions using the basic operations from existing function types studied (F.BF 1a, 1b, 2, 3, 4a).

In algebra 2, students performed basic arithmetic operations (+, -, *) with complex numbers; they used complex numbers in polynomial identities and in solving quadratic equations (N.CN.1, 2, 7).

Current Learning

Students graph polynomial functions using key features of the graph and analysis of the function. Students identify zeros of the function when suitable factorizations are available, and they identify and describe the end behavior of the function. Students use the Rational Zero Theorem, DesCartes' Rule of Signs, synthetic division, upper and lower bounds, and the Intermediate Value Theorem to find all roots of a polynomial equation.

Given roots over the set of all complex numbers, students create the equation of polynomial functions.

Students graph rational functions using key features of the function. They identify zeros of the function and identify and describe the end behavior of the function. Students also analyze the function to identify asymptotes and point discontinuities.

Students build new functions using composition of functions to relate two or more quantities and to describe the relationship between and among those functions.

Students find the conjugate of complex numbers and use them to solve polynomial equations with real coefficients that have complex solutions. Given both real and complex roots of the equation, they also find a polynomial equation.

Future Learning

Later in this course, students will apply these concepts in studying exponential, logarithmic, and trigonometric functions and their inverses.

Also later in this course, students will evaluate a limit of a function based on continuity, end behavior, and asymptotes.

All these concepts will lay the foundation for calculus.

Additional Findings

The findings in this study indicated that students' mastering the connection between graphs and equations is not as straightforward as is often assumed. It is important for students to understand which representation for a function is best. However this cannot be where a teacher stops. Developing a thorough and deep understanding of functions also means being able to interchange and work flexibly between the different representations. To help students master these connections, it is important to give students opportunities to build connections between the algebraic and graphical representations through interacting with them [Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), (pp. 500–507)].

According to *Benchmarks for Science Literacy*, studying algebra includes reflecting on relationships and patterns and their representation as data. Students have an opportunity to reflect on relationships in real-world patterns and to represent this relationship in numeric, graphical, analytical, and verbal modalities (pp. 23–25, 29, 214–216).

Teacher Notes:

Use synthetic division as a process to determine the roots of a polynomial function.

How can the characteristics of a function determine the equation of the function? (e.g., the equation of a function with the following characteristics: undefined at $x = 2$, as $x \to 3^-$, $f(x) \to \infty$; as $x \to 3^+$, $f(x) \to \infty$; as $x \to \infty$, $f(x) \to 1$, and $f(4) = 0$.)

Challenges for students include:

- Paying attention to the specificity of notation and vocabulary in the analysis of functions (e.g., in looking at extrema, students should look at function value rather than the ordered pair).

Precalculus, Quarter 1, Unit 1.3

Exponential and Logarithmic Functions

Overview

Number of Instructional Days: 15 (1 day = 45 minutes)

Content to Be Learned

- Analyze different representations of exponential and logarithmic functions to identify key features, such as domain, range, intercepts, end behavior, asymptotes, and intervals of increase and decrease.
- Classify models as exponential growth or decay.
- Create and solve appropriate exponential and logarithmic models from applications.
- Utilize appropriate, mathematically accepted notation when identifying key features and composing functions.
- Analyze the inverse relationship between exponential and logarithmic functions.

Mathematical Practices to Be Integrated

Make sense of problems and persevere in solving them.

- Explain correspondence among equations, verbal descriptions, tables, and graphs.
- Graph important features and relationships of exponential and logarithmic functions.

Model with mathematics.

- Apply the mathematics known to solve problems arising in everyday life, society, and the workplace.
- Analyze relationships mathematically to draw conclusions.

Attend to precision.

- Apply meanings of mathematical symbols consistently and accurately.
- Express numerical answers with a degree of precision appropriate for the problem context.

Essential Questions

- What would be the graphical representation of an exponential or logarithmic function?
- What real-world scenario can best be modeled with an exponential growth model? Exponential decay model? Logarithmic model?
- What is the relationship between the exponential function and the logarithmic function?

Written Curriculum

Common Core State Standards for Mathematical Content

Interpreting Functions	**F-IF**

Analyze functions using different representations

F-IF.7 Graph functions expressed symbolically and show key features of the graph, ~~by hand in simple cases and~~ using technology ~~for more complicated cases~~.★

d. (+) Graph ~~rational~~ functions, identifying zeros and asymptotes ~~when suitable factorizations are available~~, and showing end behavior.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior. ~~and trigonometric functions, showing period, midline, and amplitude.~~

Building Functions	**F-BF**

Build a function that models a relationship between two quantities

F-BF.1 Write a function that describes a relationship between two quantities.★

c. (+) Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

Build new functions from existing functions

F-BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Common Core Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Clarifying the Standards

Prior Learning

Students learned, as general principle, the concept of a function and the use of function notation in algebra 1 (F.IF.1, 2, 3). Students interpreted linear, exponential, and quadratic functions that arose in applications in terms of a context (F.IF.4, 5, 6). Students analyzed linear, exponential, quadratic, absolute value, step, and piece-wise functions using different representations (F.IF.7a, 7b, 7e, 8a, 8b, 9). Students built new functions using the four basic operations from existing linear, exponential, quadratic, and absolute value functions (F.BF.1a, 1b, 2, 3, 4a).

In algebra 2, students interpreted and used key features to guide the selection of an appropriate type of model function (F.IF.4, 5, 6, 7b, 7c, 7e, 8a, 8b, 9). Students also built new functions using the four basic operations from existing simple radical, rational, and exponential functions (F.BF.1b, 3, 4a).

Current Learning

Given different function representations, students identify domain, range, intercepts, end behavior, asymptotes, and intervals of increase and decrease.

Given different representations of exponential functions, students classify the model as exponential growth or decay.

Given real-world scenarios, students choose and solve appropriate exponential or logarithmic models.

Given exponential and logarithmic functions, students compose new functions. They analyze exponential and logarithmic functions and their inverses.

Future Learning

Later in this course, students will apply these concepts in studying trigonometric functions and their inverses. They will evaluate a limit of a function based on continuity, end behavior, and asymptotes.

All these concepts will build the foundation for calculus.

Additional Findings

The findings in this study indicated that students' mastering the connection between graphs and equations is not as straightforward as is often assumed. It is important for students to understand which representation for a function is best. However this cannot be where a teacher stops. Developing a thorough and deep understanding of functions also means being able to interchange and work flexibly between the different representations. To help students master these connections, it is important to give students opportunities to build connections between the algebraic and graphical representations through interacting with them [Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), (pp. 500–507)].

According to *Benchmarks for Science Literacy*, studying algebra includes reflecting on relationships and patterns and their representation as data. Students have an opportunity to reflect on relationships in real-world patterns and to represent this relationship in numeric, graphical, analytical, and verbal modalities (pp. 23–25, 29, 214–216).

Teacher Notes:

Although it is currently not in the Common Core State Standards, time may need to be spent on the properties of logarithms, as they will be used to integrate in calculus.

Potential Application Example:

> After researching your dream car, comparing loan rates and methods of compounded interest, what would be the best deal for you as the purchaser? What would be the best deal for the loan holders?

Challenges for students include:

- Paying attention to the specificity of notation and vocabulary in the analysis of functions (e.g., in looking at extrema, students should look at function value rather than the ordered pair).
- Determining domain for composition of functions.